A critique of the econometrics of happiness:  
Are we underestimating the returns to education and income?  

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Abstract  
A long-overlooked problem exists with numerical, subjective response data such as that which underlies the economics of happiness. Addressing this problem resolves a longstanding and unexplained anomaly in the life satisfaction literature, namely that the returns to education, after adjusting for income, appear to be small or negative. I also show that, due to the same econometric problem, the marginal utility of income in a subjective wellbeing sense has been consistently underestimated by economists of happiness. The problem is a tendency by some respondents to simplify the response scale by considering only a subset of the possible responses. This renders even the weak ordinality assumption used with such data invalid, in principle. First, I use a multinomial logit model on subjective wellbeing data to diagnose the factors associated with “focal value response” (FVR) behavior. After empirically characterizing this conspicuous problem, I introduce a new computational approach to estimate the bias resulting from FVR, to correct estimates of univariate and multivariate inference, and to estimate an index quantifying the fraction of respondents who exhibit FVR. I show that large biases are possible to mean life satisfaction and to marginal effects, and that, empirically, significant downward biases in estimates of the benefits of education and income are ubiquitous in this literature. Lastly, I provide some heuristic tools to check for sensitivity to FVR, and suggest innovations needed to better measure subjective wellbeing.

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1 Introduction

Now firmly entrenched in the economics literature, in national statistical agency data collection, and in the dialogue about progress and well-being\(^1\), survey-based subjective evaluations of life are the basis for evaluations of welfare benefits and costs of everything from inflation and unemployment to air pollution to being married. Estimates of the psychological benefit of increased income, using this approach, are at least four decades old, and those evaluating the net individual return of additional education have been carried out for at least three decades. In terms of optimally allocating human resources, not much could be more central than knowing the marginal utility of income and education.

However, inferred valuations of income and education using life satisfaction have been consistently low. Moreover, at first look, subjective evaluations of life appear to be an unlikely candidate for the economist’s arsenal. Their coherence and value rely on a series of considerable cognitive tasks to be performed in short order by the respondent. When asked, “Overall, how satisfied are you with life as a whole these days, measured on a scale of 0 to 10?” a respondent must in some sense\(^2\) (i) conceive of the domains, expectations, aspirations or other criteria salient to her sense of experienced life quality or satisfaction; (ii) assemble evidence pertaining to each ideal, such as recent affective (emotional) states, significant events, and objective outcomes; (iii) appropriately weight and aggregate this evidence according to its importance to overall life quality, and (iv) quantify the result according to the somewhat arbitrary, discrete scale specified in the question.

This is without doubt a tall order, and any embrace of life satisfaction data rests on their remarkable reproducibility and apparent cardinal comparability, possibly along with the principle that any objective indicator of experienced well-being must ultimately be accountable to a subjective one. While various studies have sought, with limited success, to find differences in interpretation of the question or norms of expression across cultures and languages (Helliwell et al., 2010; Exton, Smith, and Vandendriessche, 2015; Lau, Cummins, and Mcpherson, 2005; Clark et al., 2005), an important fact is that, uniformly across cultures, responding to the question is cognitively demanding. This paper focuses specifically on the consequences of heterogeneity across respondents in their quantitative ability relevant to the final one or two stages in the process outlined above. The crux is that people with less facility with numbers may simplify the numerical response scale for themselves. In particular, I will show that they do this by restricting the set of numerical response options under consideration. This can be expected to introduce complex biases in mean life satisfaction and in estimated marginal effects on life satisfaction, in particular with respect to education and other correlates of numerical literacy itself.

In this paper I give previously unreported evidence of the prevalence and quantitative significance of this problem with the measurement of life satisfaction and other similar numerical scale measures, which underlies the entire field of the “economics of happiness.” While most studies justify a cardinal interpretation of the response scale in the life satisfaction question, and at least an ordinality assumption is universal, the “focal value response” (FVR) behavior described here violates even ordinality of response options.

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\(^1\)The life satisfaction question and close cousins such as the Cantril Ladder question are posed in numerous national and international social surveys, both cross-sectional and panel.

\(^2\)Typically, cognitive evaluations of life consist of a single, subjective, quantitative question like this one, and responses are used directly as a cardinal or ordinal proxy for welfare, i.e., “experienced” utility.
However, estimating the size of systematic biases associated with inference based on SWL reports requires a model accounting for the phenomenon. I present such a model and show why biases on estimates can be large or small and positive or negative, and why most of the literature to date has significantly underestimated the experienced benefit from both education and income. I also define a new index (the Focal Value Response Index, or FVRI) and estimator for the fraction of respondents in a sample who have chosen an alternate, simplified response scale.

After validating the estimation approach using synthetic data, I apply it to make new estimates of the relationship between education, income, and wellbeing using social survey data from Canada.

Lastly, I re-analyze representative studies in the literature to show that two previously anomalous but reproducible findings — that a disadvantaged population reports high life satisfaction, and that the return to extra years of education after primary school are net negative — are overturned when taking into account focal response behavior.

Unlike some previous critiques of the econometrics of life satisfaction (e.g., Bond and Lang, 2019), this study is focused not on the outside possibilities of what could be alternative interpretations of wellbeing data, but rather the most likely and easy-to-understand interpretation of conspicuous, empirical, stylized facts. The implications are concrete and calculable rather than speculative. Because a number of governments are gearing up to carry out detailed cost/benefit analysis using cognitive life evaluation data and regressions as key evidence (Frijters et al., 2020; Frijters and Krekel, 2021; Happiness Research Institute, 2020; Grimes, 2021; Department of Finance, 2021), the ramifications for the measurement and interpretation of life satisfaction data are extensive.

The rest of this paper proceeds as follows. The remainder of the Introduction reviews some paradoxical stylized facts related to education and wellbeing in the happiness literature, explains the origin of modern survey questions like SWL, and provides evidence that respondents make costly efforts to express the precision of their answer to such questions. Next, I will convince the reader that there is a measurement problem with quantitative, subjective scales like SWL that is conspicuous, ubiquitous, and strongly correlated with educational attainment (Section 1.1) and that it has a natural explanation (Section 1) supported by the behavioral evidence. Then Section 3 presents the formal model in which a mixture of high- and low-numeracy respondents treat the scale differently. Section 4 validates the computation approach using synthetic data, and explores the complexity of biases that can result from FVR. Section 5 presents the main empirical estimates of the ubiquitous biases in the happiness literature, while Section 6 gives two examples of necessary reconsideration of previously published findings. Some concluding thoughts are in Section 7.

1.1 Well-being effects of education and income

The economic interest in subjective wellbeing measurement arises from its use as an empirical proxy for general welfare, at least in a marginal sense. The most prominent findings from the economics of happiness have, since the 1970s, related to the surprisingly small coefficients (marginal effects) for income (e.g., Dolan, Peasgood, and White, 2008; E., 2018; Deaton, 2008; Diener et al., 1993; Duncan, 1975; Easterlin, 2013, 1995; Mikucka, Sarracino, and Dubrow, 2017) as well as education. Education may be expected to confer benefits not just through higher income but also through better health behaviors and enhanced social cap-
ital of various forms (e.g., Helliwell and Putnam, 2007), as well as something more abstract, which may be thought of as psychological capital, providing direct benefits or complementing other consumption. However, among the more surprising stylized facts is that formal education does not help much to explain SWL once income is accounted for. This generalization hides considerable variation in the literature; since the various channels and directions of influence are not easily identified, and the relationship may not even be monotonic (Stutzer, 2004), estimates vary from slightly positive to slightly negative overall effects of having extra education.

Similarly, although the literature on the importance of income and income growth on SWL is enormous and involves a large potential role of consumption externalities (Barrington-Leigh, 2014), one may summarize the findings of SWL research by saying that income has been found to be less of the story than economists might have anticipated (e.g., Hamilton, Helliwell, and Woolcock, 2016; Blanchflower and Oswald, 2004). Notably, there is less surprise to other disciplines of human nature who have traditionally understood human wellbeing and behavior in terms of social roles, norms, and intrinsically valuable interactions. In any case, the large compensating differentials found for having positive social relationships (trust, engagement, meaningful work, friendships, intimate relationships, etc; see for example Powdthavee, 2008; Helliwell and Barrington-Leigh, 2011; Helliwell and Putnam, 2004) reflect a small denominator, i.e. the value of income for increasing life satisfaction. Thus the reasonable argument that increased income ought to allow for more choice about the pursuit of other, more social, supports of SWL faces the empirical evidence that such a trade-off is apparently not being generally realized for some reason.

This paper does not aim to explore all the reasons for this well-established evidence about quality of life from subjective response data. Instead, it characterizes a measurement error in which those with lower education, and possibly lower income, may be more likely to under-utilize the SWL response options in such a way that tends to enhance (upwardly bias) their reported life satisfaction.

1.2 Evolution of precision in subjective, quantitative reports

The history of survey questions on subjective assessments mirrors in part technological norms. Early innovators in monitoring subjective well-being in social and household surveys tended to use a four point or five point scale, typically with Likert-like bipolar verbal response options. In such questions, exemplified in Figure 1, the numbers were not meant as cues for the respondent. Naturally, the variation, or precision, was limited because most respondents chose one of the top two options (see Figure 2). As limitations of paper survey media have been erased by the adoption of computer aided interviews, these subjective scales have expanded their resolution. However, with more than five or seven response options, verbal cues are typically not provided except for the highest and lowest response options. Responses instead become numerical. For instance, after many years of asking satisfaction with life (SWL) questions with a variety of scales (Figure 2), Statistics Canada settled over a decade ago on a particular wording with an

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3Studies routinely control for current income rather than wealth when assessing SWL benefits of education. Current income may be an especially poor proxy for lifetime income in this context because choosing to pursue extra education entails a trade-off between short-term income and future income. This leaves educational attainment as a positive proxy for unmeasured future income expectations, making the low coefficients measured on education even more surprising.
11 point scale.

The OECD (2013) has also developed recommendations for standardizing the way such questions are asked by all national statistical agencies. The *de facto* standard for SWL now is an 11-point scaling, from 0 to 10, with the lower extreme meaning, for example, “completely dissatisfied,” the upper signifying “completely satisfied,” and the interpretation of the remaining values left up to the respondent.

1.3 Why respondents choose focal values: behavioral evidence

There have been plenty of studies on the optimal number of response options in survey questions with verbal cues for each option. For instance, it may be that in an oral interview, i.e. with no visual cues, four or five responses are the maximum that can be handled without confusion or overload (Bradburn, Sudman, and Wansink, 2004). When the scale is explicitly numeric, as with modern SWL measures, there also arises a trade-off between the cognitive load imposed by a scale and the precision it allows.

From the respondent’s point of view, this trade-off is between *expressive capacity*, i.e. allowing for precise responses in order to capture variability among respondents and over time, and *processing capacity*, i.e. allowing for the cognitive steps described earlier to occur without overburdening the respondent. Overburdening would result, at best, in the respondent not fully optimizing her answer or not properly interpreting or using the given range of responses (OECD, 2013). Various studies on this balance have tended to favor 11-point quantitative scales over coarser option sets (e.g. 7-point scales) as well as over nearly continuous options (Alwin, 1997; Kroh et al., 2006; Saris, Van Wijk, and Scherpenzeel, 1998; OECD, 2013). This literature has mostly focused on the optimal number of responses when each response has a verbal cue, while in this paper the focus is on numerical precision in scales with purely numeric options. That is, in this paper, SWL questions all have a scale that is internally defined by the respondent, except for the two end-points, which have labels such as “completely satisfied” and “completely dissatisfied.” Beyond these two conceptual values, the task is therefore particularly quantitative, especially when there are as many as 9 or more unlabeled numerical options.

As a final piece of empirical motivation for the modeling approach developed below, I present in Figure 3 the distribution of responses to a life satisfaction question which accommodated a higher resolution than the canonical 11-point format. These data come from an
Fig. 2. Histograms of SWL responses over time in Canada
online “vignette” survey, in which respondents predict the life satisfaction of a hypothetical individual based on a short description of her life circumstances. The only relevant feature to note here is that, while a precision of 0.1 on a 0–10 scale is offered to the respondent, it appears, based on the integer and half-integer focal values, that different individuals have simplified the scale for themselves. The interface was a slider which gave no preference for any particular values; in other words the cost to the respondent of choosing a non-integer answer over an integer answer were purely cognitive. Indeed, if a respondent had a particular precision in mind, greater than 0.1, then the existence of the observed focal values indicates that extra effort was exerted in order to leave the slider precisely on a half- or whole-integer value. This can be interpreted as evidence of an effort to faithfully communicate a mental result, motivated solely by the wish for self-expression.

2 Predictors of focal value selection

The motivation for this paper is the observation that histograms of 10-point and 11-point SWL responses are multi-model, often having not one but four peaks. Figure 2 shows the distributions from a number of surveys in Canada. While the most common response is usually “8,” supplementary enhancements are clearly visible at 10, 5, and the lowest value, 0 or 1. I refer to these as “focal values.” Remarkably, these conspicuous focal value enhancements have been almost entirely ignored in the decades of SWL literature so far.

A small number of studies have at least remarked on the focal value issue, but without a full account or explanation. Landua (1992) analyses response transition probabilities in the German Socio-Economic Panel, and Frick et al. (2006) confirm his report that respondents have a tendency to move away from the endpoints over time. In fact, this could be driven largely by the focal value behavior diminishing as panel participants, especially those with low numeracy, gain familiarity and comfort with the scale. Dolan, Layard, and Metcalfe (2011) mention that SWL ratings in one study are positively associated with life circumstances as one would expect, except at the top of the scale, where “those rating their life satisfaction as ‘ten out of ten’ are older, have less income and less education than those whose life satisfaction is nine out of ten.” They speculate a reason unrelated to cognitive limitations for this observation.
but declare that “This issue warrants further research”. The present paper may fully resolve this mystery.

In order to characterize focal value response behavior as it relates to education, and to characterize the problems it causes for inference, I focus in this section, and again in Section 5, on one cycle from the Canadian Community Health Survey, an annual cross-section which includes an 11-point life satisfaction question, as well as educational attainment recorded in four categories: less than high school graduate, high school graduate, some post-secondary training, and completed post-secondary training.\textsuperscript{4} Relatively few respondents respond with the third category.

Breaking up the SWL response distribution according to educational attainment (Figure 4) reveals a striking feature. Qualitatively, we can say that the focal value enhancements decrease monotonically with increasing education, and in the highest education category two of the extra peaks are absent, leaving only a unimodal distribution around SWL=8 along with a slight enhancement at SWL=0. This suggests that, in addition to any possible association between education level and some internal subjective wellbeing, there is a difference across education levels in the way that satisfaction is translated or projected onto the response

\textsuperscript{4}In the 2011--2012 cycle, out of a total of 124929 respondents, 96937 reported their age as at least 25 years, and also answered the SWL and educational attainment questions.
options. Mean SWL is shown in each panel of Figure 4. Those with the most education are happier than those with the least, but as I shall show below, the estimated relationship between SWL and education in this sample is negative after controlling for income.

2.1 Modeling the full response distribution

The observations above are clear evidence of focal value behavior that is related to educational attainment, possibly among other determinants. This implies that SWL response scales cannot, in principle, be assumed to be ordinal. That is because someone with lower education, but all else equal, may as a consequence experience lower life satisfaction but also be inclined to report a higher value due to rounding up from a 3 or 4 to a 5, or from a 8 or 9 to a 10.

Traditional methods used in econometric inference from SWL — such as OLS, ordered logit, and ordered probit, and related time series and instrumented analogues — leverage strong assumptions about the symmetry of effects of explanatory variables on each step of the response scale, as well as assuming cardinality or at least ordinality among response values. Those specifications are not flexible enough to account for the heterogeneous influence of predictors like education on focal and non-focal response values.\(^5\)

An alternative approach is to relax fully the ordinality assumption for response options, and model the probability of each response independently, subject only to the constraint that the probabilities add up to one. The standard multinomial logit model\(^6\) does this, but has not so far been applied to SWL data. For current purposes, multinomial logit estimations provide an excellent diagnostic tool for finding predictors of focal value behaviour; however, a more structured approach described later will be required to estimate biases resulting from such reporting behavior.

Figure 5 shows marginal effects of education and income on response probabilities of each of the 11 points in the SWL scale, from a multinomial logit model using education, logarithmic income, age, and age\(^2\) as predictors. Under an ordinality assumption, one might expect marginal effects to rise monotonically with response value, since a better circumstance like education or income should lead to an increase in the relative probability of response \(s + 1\) as compared with response \(s\). Indeed, other than the focal response values 0, 5, and 10, the marginal effect of one step higher educational attainment (for instance, graduating from high

\(^5\)It is important to note that, unlike OLS, ordered logit and ordered probit can fully account for the multi-peaked distributions shown in Figure 4. That is due to the flexibility given by the cut points in those models, which can squeeze together or stretch apart for a given response option in order to decrease or increase (respectively) the option’s predicted response rate.

\(^6\)The multinomial logit model, in its latent variable formulation, consists of a system of equations generating scores \(Y^*_i,j\) for each individual \(i\) and response option \(j \in \{0 \ldots 10\}\):

\[
\begin{align*}
Y^*_i,0 &= \beta_0 \cdot x_i + \varepsilon_0 \\
Y^*_i,1 &= \beta_1 \cdot x_i + \varepsilon_1 \\
&\quad \ldots \\
Y^*_i,10 &= \beta_{10} \cdot x_i + \varepsilon_{10}
\end{align*}
\]

where \(\varepsilon_j \sim EV_1(0, 1)\), i.e. the error terms have standard type-1 extreme value distributions. The predicted response for individual \(i\) is the value of \(j\) with the highest \(Y^*_i,j\). Probabilities of each response for each individual are fully determined if one of the above equations, considered a “base case,” is dropped. This model is, strictly speaking, also misspecified for SWL data, since the formal independence of irrelevant alternatives (IIA) assumption is violated by the numbered options of an SWL question. The IIA requirement is frequently overlooked in applications of the multinomial logit.
Fig. 5. Marginal effects of education and income on individual response probabilities. Multinomial logit estimate of individuals’ probability of giving each possible response to the life satisfaction question. For education, marginal effects show a monotonic pattern with increasing response value, except for the remarkable outliers at 0, 5, and 10. These indicate that education and, simultaneously but to a lesser degree, income are significant predictors of the tendency to use a simplified response scale. Error bars denote 95% confidence intervals.

school) is weakly increasing by reported SWL. By contrast, the effects on the focal value responses are, with high statistical significance, negative\(^7\) outliers lying far below what would be expected based on the pattern of adjacent values. They show that more education significantly reduces the probabilities of each focal value response. While precisely estimated, these effects are not overwhelming; the difference between the marginal effects for \(P_{SWL=9}\) and \(P_{SWL=10}\) is \(\sim 3\%\). By contrast, overall the raw data suggest \(P_{SWL=10}\) is 19\% higher than \(P_{SWL=9}\), and the standard deviation of the education variable is 1.2. The corresponding anomaly for the marginal effect of increasing income is another \(\sim 2\%\) between \(P_{SWL=9}\) and \(P_{SWL=10}\) for each unit increase in log income. Thus, according to this rather loose accounting, this estimate may not explain the entire focal value enhancement observed in the response distribution. This is not surprising, as formal education and — even more so — income can only be expected to be weak proxies for numeracy or cognitive capacity of the kind needed to deal with the 11-point response scale.

In fact, without a more structural model, it is difficult to interpret the effects in Figure 5, even though they show a convincing pattern, because there is no \textit{a priori} expectation about trends in marginal effects as a function of successive response values. This is because there may be complex correlations between covariates. Indeed, relaxing the ordinality assumption across response options is the motivation for using a multinomial logit model in the first place.\(^8\)

\(^7\)For each explanatory variable, the sum of all marginal effects on probabilities is zero by construction.

\(^8\)If the model is estimated using only the education variable as a predictor, the results show essentially the same pattern, with even stronger effects for the focal values. However, in that case, it is also possible
3 Cognitive mixture model

Motivated by the evidence above, I interpret the enhanced use of focal values as an indication that respondents have simplified their cognitive task by coarsening the numerical scale. In principle, this could be due to the challenging nature of any of the cognitive steps outlined in the first paragraph of Section 1; however, because of its evident inverse association with education, I infer that it reflects a lack of something like numeracy which can be trained, rather than a difficulty inherent to the intrinsic awkwardness and subjectivity of evaluating one’s life in an all-encompassing way. Accordingly, I assume that cognitive processes of individuals differ only in the execution of step (iv). This step is broken down into:

(a) Choose a response resolution to use, either:

1: the full scale or
2: a subset of the scale, consisting of the top, bottom, and a central value.

(b) Project the result of step (iii) of Section 1 onto this discrete, quantitative scale.

The discrete choice in (a), above, is modeled as dependent on proxies for numeracy, without any explicit consideration of costs and benefits to the decision maker. Conceptually, the individual benefits of using the full scale are self-expression and performing one’s best at fulfilling the purpose of the survey; the costs are those of the cognitive task involved (see Section 1.3).

Essentially, I assume there are two types of respondents, high numeracy and low numeracy, who use the full scale $\mathbb{S} = \{0, 1, \ldots, 10\}$ and the focal value subset $\mathbb{F} = \{0, 5, 10\}$, respectively. Figure 6 shows a diagram of the assumed causality in this picture.

The choice, or categorization, of high or low numeracy is modeled probabilistically at the individual level. That is, given individual characteristics $\mathbf{x}$, the model specifies a probability $\Pr(\text{high} | \mathbf{x}) = 1 - \Pr(\text{low} | \mathbf{x})$ of the individual having high numeracy, and thus responding using the full scale. In addition, the probability of each individual giving each discrete response to the life satisfaction question is modeled for each of the two choices of scale, i.e., for each of the two numeracy types. This forms a “mixture model” in which the overall probability of an individual giving a particular response $s \in \mathbb{S}$ to the SWL question is modeled as a function of individual characteristics $\mathbf{x}$ by appropriate aggregation of these constituent probabilities:

$$\Pr(s | \mathbf{x}) = \Pr(high | \mathbf{x}) \Pr(s | \mathbf{x}, \text{high}) + \Pr(low | \mathbf{x}) \Pr(s | \mathbf{x}, \text{low})$$

(2)

The model is similar to the ordinal-outcome “finite mixture model” of Boes and Winkelmann (2006) except that here the mixing probabilities are also dependent on individual characteristics (see also Everitt and Merette, 1990; Everitt, 1988; Uebersax, 1999).
Fig. 6. Directed Acyclic Graph (causal diagram) for the mixture model. Experienced utility $S^\star$ is contemporaneously determined by some life conditions (such as education) $H$, which also affect numeracy $N$, and by other life conditions $Y$. The numerical self-report $S$ is a reflection of $S^\star$ but is influenced by numeracy through the focal value effect. The contemporaneous conditions $Y$ and $H$ may be codetermined by prior influences $Z$. The boxed variables are not directly observable.

We may motivate the model by conceiving of two latent variables. One, $S^\star$, is the experienced well-being we wish to measure. That is, the respondent’s internal, weighted evaluation of life has an explicit form, $S^\star$, in the model as a continuous latent variable which depends on life circumstances. Another, $N^\star$, is the continuous measure of numeracy, predicted by education level and possibly other individual characteristics, which determines whether a respondent will project $S^\star$ on to the full scale provided, or simplify it to a 3-point subset of focal values. The response process then occurs in three steps: (i) the respondent carries out all but the last step described in Section 1 to arrive at $S^\star$; (ii) the respondent chooses whether or not to simplify the scale, effectively eliminating certain response options; and based on this choice, (iii) the respondent carries out the final cognitive step of projecting $S^\star$ onto the quantitative scale in the questionnaire, resulting in their observed response, $s \in S$.

For each type, high and low numeracy, the possible responses represent an ordered set, and response probabilities can be modeled using an ordered logistic or ordered probit formulation. I assume that the latent variable $S^\star$ is the same for these two parts of the model, i.e., I assume that the internal well-being measure for high and low types exhibits the same dependence on other individual characteristics. Thus, types differ only in their reporting behavior. Altogether, then, the parameters to be estimated are the coefficients and cutoff value predicting the numeracy classification; the coefficients predicting the latent well-being variable $S^\star$, and two sets of thresholds used to transform $S^\star$ into discrete values in the focal-value or full-range ordinal scales.

Formally,

$$
\begin{align*}
\Pr (s \mid x) &= \Pr (\text{high} \mid x) \Pr (s \mid x, \text{high}) + \\
&\quad [1 - \Pr (\text{high} \mid x)] \Pr (s \mid x, \text{low}) \\
= &F_N (z' \beta_N) \left[ F_S (\alpha^H_s - x' \beta_S) - F_S (\alpha^H_{s-1} - x' \beta_S) \right] + \\
&\quad [1 - F_N (\alpha^N - z' \beta_N)] \times \\
&\quad [F_S (\alpha^L_s - x' \beta_S) - F_S (\alpha^L_{s-1} - x' \beta_S)]
\end{align*}
$$

(3)

for each value $s \in S$. Here $x$ is the full vector of observed explanatory variables, used to predict well-being, while $z$ is a vector of a possibly-overlapping set of observed variables, used to predict numeracy. There is a column of constants included in $z$, but none in $x$. Twelve values of $\alpha^H_s$, with $\alpha^H_{-1} = -\infty$ and $\alpha^H_{10} = +\infty$ and $\alpha^H_s > \alpha^H_{s-1} \forall s \geq 0$, are the thresholds for
responses by high types; and four distinct values of $\alpha_L$ are those for low types, with $\alpha_{L_{-1}} = -\infty$, $\alpha_{L_0} = +\infty$, $\alpha_{L_1} = \alpha_{L_2} = \alpha_{L_3} = \alpha_{L_4}$, and $\alpha_{L_5} = \alpha_{L_6} = \alpha_{L_7} = \alpha_{L_8} = \alpha_{L_9}$.

The probabilities come from the standard ordered regression formulation; with cumulative distribution function $\Phi(\cdot)$,

$$
P(S^* > \theta_i | x) = P(x' \beta + \varepsilon > \theta_i) = P(\varepsilon > \theta_i - x' \beta) = 1 - \Phi(\theta_i - x' \beta)
$$

Eq. (3) is a rather flexible specification in that the two sets of SWL cutoff values for low and high types are determined independently of each other. Rather than allowing for a separate set of two thresholds for the ordinal value cutoffs, a possible simplifying assumption is that the collapsing of the 11-point scale to a 3-point scale occurs precisely where one might expect, e.g., $\alpha_{L_4}$ should occur somewhere between $\alpha_{H_2}$ and $\alpha_{H_3}$. For simplicity, this assumption is

$$
\begin{align*}
\alpha_{L_0} &= \frac{\alpha_{H_2} + \alpha_{H_3}}{2} \\
\alpha_{L_5} &= \frac{\alpha_{H_7} + \alpha_{H_8}}{2}
\end{align*}
$$

Reassuringly, I find below that leaving the cut-points independently flexible results in estimates which approximate this assumption.

### 3.1 Identification

Before proceeding, we must discuss whether the parameters in this model are identified in principle. Identification of this model is a challenge because in general the same predictors $x$ may be used to predict the latent numeracy variable and to predict the latent well-being variable. As a result, there may be more than one set of parameters which equally well explains observations for a given population. An exclusion restriction to ensure sufficient independent variation in the predictors for numeracy — for instance, by excluding the columns of $z$ from $x$ — would overcome this problem. However, as is typically the case when a broadly-scoped SWB is the dependent variable, there is usually nothing that can be excluded as an influence. Moreover, a particular interest motivating this study is to assess the bias on estimates of the well-being effect of education, which is the primary available predictor of numeracy.

Alternatively, a thin set identification (often, “identification at infinity”) could be possible if, for instance, those with very high or very low education (or other $z$ predictor) were nearly certain to be the numerate or non-numerate type, respectively. A predictor with large support

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10In the case of the ten-point (1–10) SWL scale, there is one fewer distinct $\alpha^H$ value, and the corresponding restriction is:

$$
\begin{align*}
\alpha_{L_1} &= \frac{\alpha_{H_3}}{2} \\
\alpha_{L_5} &= \frac{\alpha_{H_7} + \alpha_{H_8}}{2}
\end{align*}
$$
would be a sufficient condition for such certainty under the specification 3. One standard problem with this kind of identification is that it relies on an assumption of a uniform effect of the covariate across its support (i.e., that the specification in 3 is correct) in order to produce an unbiased estimate. By contrast, the uniformity of canonical coefficients in estimates of SWL may be in doubt because it does not hold across different values of SWL. This is shown in the Appendix. In addition, the covariates in the present case have only a few discrete values and therefore do not have large support.

A simpler approach relies on the fact that according to Eq. (3), respondents who do not give a focal value response to the SWL question are necessarily of the high type. Therefore, the covariates to be estimated for predicting the latent SWB value $S^*$ are point identified by restricting the sample to the subset of high types who did not respond with a focal value. In fact, because no “5’s” are observed in this group, it consists of two subsamples: those with observed $s \in \{1, 2, 3, 4\}$ and those with $s \in \{6, 7, 8, 9\}$. An analysis along these lines is given below, but it again indicates that the assumption of constancy of these effects is violated. Indeed, in principle if the effects were constant, the coefficients for $S^*$ could be estimated separately on each of the six edges between SWL values that do not include the focal values. For instance, using the sample of all respondents who answered either 1 or 2, the probability of choosing the higher response could be modeled using a logit (or probit) estimator. Again, this is carried out in the Appendix, for the purpose of demonstrating another heuristic for detecting focal value response behavior, and putting its magnitude into context.

With these caveats, consider the identification of a model like Eq. (3) using a logistic distribution function (with $\sigma = 1$ for $F_N(\cdot)$), to model the probability of an individual being numerate, and the same distribution function for $F_S(\cdot)$ to model the probabilities of discrete responses for both low and high types. That is,

$$F_S(t) = F_N(t) = \frac{1}{1 + e^{-t}}$$

The mixture model then comprises a logit estimator for the numeracy type and ordered logit specifications for each of the two discrete response scales. The same coefficients $\beta_S$ are used in the high-type and low-type terms of Eq. (3) because they represent a common latent well-being value $S^*_i = x_i^T \beta_S$.\footnote{An epistemological issue is that the measure of interest for welfare analysis is now the estimate of a latent well-being value $S^*$, which is not observable. This sounds somewhat undesirable from an empiricist’s point of view. However, it also arguably much less undesirable than using the latent variable of cardinal utility to do welfare analysis, as is customary in the revealed preference approach which still dominates most welfare analysis. The empirical accountability of the methods described here comes from testing how much more successful a model which takes into account focal value behaviour is than one which does not. By contrast, there is no empirical accountability to the use of revealed preference methods for welfare analysis, once a rationality assumption is relaxed. The imputation of a “correct” SWL value for someone who reports a focal value is also in principle falsifiable, since one can imagine helping to familiarize a respondent with the full response scale until they are comfortable giving a higher resolution answer to the SWL question.} No constant term is included in $S^*$, so observed responses from the high types identify all ten thresholds $\alpha_H$ in a normal ordered logit model. Moreover, the focal-value SWL responses also fix both thresholds $\alpha_L$ for the low type probabilities, and one should expect these thresholds to be numerically consistent with the $\alpha_H$, i.e. to lie appropriately in between the higher-resolution thresholds. Although the thresholds governing responses 5 and 10 may be less obviously uniquely identified due to the interplay of the thresholds for high and
low types, and due to the overlap between vectors x and z which determine both the mixture and the latent well-being, there is enough structure in the model to separate out these degrees of freedom.

Computation of the mixture model was carried out using the no-U-turn sampler (NUTS) variant of a Hamiltonian Monte Carlo (HMC) algorithm, which is in turn a Markov Chain Monte Carlo (MCMC) method (Stan Development Team, 2018; Riddell, Hartikainen, and Carter, 2020; Carpenter et al., 2017) and handles the non-convex optimization well. Section A.1 provides more detail on estimation, including analytic derivations of gradients and Hessian for a log likelihood approach.

3.2 Theoretical maximum bias

Before proceeding to some representative simulations and empirical estimates, it is possible to make statements about the maximum bias resulting from focal value response behavior. Consider a population with a narrow distribution of “true” life satisfaction, with mass concentrated just above 7.5 on the 10-point scale. For simplicity, consider only two levels of education, high and low, which have no effect on life satisfaction but which fully determine focal value behavior. In this case, low-type respondents will report a “10” while high type respondents will report an 8. Thus, while the true marginal benefit of education would be 0, an OLS (i.e., linear model) estimate would report a value of -2 SWL points on the 11-point scale, per education step. Similarly a hypothetical distribution concentrated just below 7.5 would result in a bias of +2, and another pair of extreme biases could occur around mean SWL=2.5.

Thus, the focal value problem is capable in principle of accounting for anomalous negative estimates of the value of education, but it is equally capable of explaining positive biases. Empirical evidence, given after the next section, is more interesting than principled possibilities.

4 Simulation

Before estimating the model on survey data, this section uses simulated data to accomplish two objectives. First, it validates the computational approach (Appendix A) used for estimating the mixture model, and demonstrates successful identification of simultaneous, independent effects of a predictor on numeracy and on SWL. Secondly, it provides an exploration of the complexity and scope of possible biases, due to FVR, in traditional life satisfaction estimates of means and of marginal effects. To reiterate, this “bias” is, conceptually, the difference between naively estimated values and those which would be obtained in the counterfactual case that all respondents had used the full numeric response scale, i.e. were of high numeracy type. This counterfactual can be perfectly simulated using synthetic data, but is of course unobservable in traditional empirical data.

The key potential biases to study are (1) a bias in estimated mean SWL; (2) a bias in estimated marginal effect of a variable correlated with numeracy that is used to predict both numeracy and SWL; and (3) a bias in estimated marginal effect of a variable correlated with numeracy that is used to predict SWL but is excluded from the equation for numeracy. For instance, education may be correlated with income, and both may be relevant to life satisfaction, while only education is a causal predictor of focal value behavior. Here, I generate synthetic data using the simplest model that captures these effects.
4.1 Synthetic Data Generating Process

Based on the description above, it is therefore sufficient, and minimal, to consider only two predictors of life satisfaction, which I will call education level \( h \) and income \( y \). In the formulation given in Section 3,

\[ x = (h, y) \]

with just one predictor of numeracy, namely education:

\[ z = h \]

A simple data generating process for these variables is as follows. Observations (individuals) are characterized by \( h \) and \( y \). These variables are drawn as follows:

\[ h = \mathcal{N}(0, 1) \]
\[ y = \chi \cdot h + [1 - \chi] \mathcal{N}(0, 1) \]

where the \( \mathcal{N}(0, 1) \) represent separate i.i.d draws from a normal distribution, and \( \chi \) captures the correlation between education and income. As described in Section 3, the log odds of an individual being high numeracy type (\( N=1 \)) is related to \( h \) by a logit function, viz

\[ \log \left( \frac{P(N = 1)}{P(N = 0)} \right) = \beta_{h}^{N} + \beta_{h}^{I} h \]

The numeracy type is not observable.\(^{12}\) Another non-observable is the latent response value \( S^* \), which corresponds to the experienced wellbeing which is of normative (i.e., policy) interest. It is a function of \( h \) and \( I \),

\[ S^* = \beta_{h}^{S} h + \beta_{I}^{S} I + \varepsilon_{S} \]

and is constrained to have the same coefficients for high and low numeracy types.

Observed dependent variable \( S_H \) values are integers, distributed as an ordered logit distribution with 10 cut points,

\[ S_H = \text{ologit}_{11}(S^*) \]

but these responses are only observed from high-type respondents. Similarly, low-type respondents project \( S \) onto just the three focal values, which we can express with

\[ S_L = \text{ologit}_{3}(S^*) \]

For consistency, we assume for generating data that the low-type cut points are fixed in the natural way relative to the high-type cut points, as in Eq. (4), although in estimates I will focus on the more flexible model, which allows for them to diverge.

\(^{12}\)In fact, one could rule out low numeracy for any respondent who responds with a non-focal value; however, this model eschews conditioning on the dependent variable.
4.2 Focal Value Response Index

In addition to characterizing the biases of various estimates, there is another intuitive way to quantify the problem addressed in this paper. In fact, as shown below, biases on some estimated values may be positive, negative, or zero even when FVR behavior is prominent and biases are significant for some other estimated coefficients or means. Therefore, to express straightforwardly the magnitude of the numeracy problem in a sample of respondents, the Focal Value Response Index (FVRI) is an estimate of the fraction of respondents who have restricted their answer to a set of focal values — i.e. the estimated fraction of low types.

Because numeracy, in this sense, is not observed, the FVRI is necessarily model-dependent. In the synthetic data exploration described next, the model is perfectly specified, in that observed education level is a good predictor of focal response behavior, and focal response behavior is well described as in the cognitive model described earlier. In this case, FVRI is well identified and precisely estimated. Later, in Section 5, FVRI will also be estimated for survey data.

4.3 Counterfactual SWL distribution

In addition to estimating a FVRI, and characterizing biases on estimated means and coefficients, an estimate of the mixture model can also be used to “correct” the distribution of SWL to that which would have been reported had respondents all used the full scale. Just as for mean SWL, the estimated correct distributions are compared with the simulated observations on which they are based.

4.4 Synthetic data validation results

Synthetic data and corresponding estimates were carried out for sample sizes of 300, 5000, and 20,000. Varying $\chi$, $\beta_N$, $\beta_S$, and $\alpha^H$, which are parameters of the synthetic data generation process, reveals the possible scope of biases for plausible distributions of SWL. In all cases, the mixture model correctly reproduces the parameters of the data generating process, in particular the effects of education and income on life satisfaction (zero bias in green curves of Figures 7–9). In addition, the estimates of the Focal Value Response Index agree with the true fraction of low types in the synthetic data. The same model, when estimated with a constraint that there are no low types, agrees well with a traditional ordered logit estimator.

The top left panel in Figure 7 shows the simulated observations for one case in which low-numeracy types constitute 17% of respondents. The distribution of true (latent) SWL is centered around the focal value of 5.0. As a result, the low-type responses, which are nearly all “5”, consist of equal numbers of down-shifted and up-shifted values. The naive mean of the combined distribution of SWL is therefore close to the mean of true SWL in the synthetic sample. For relatively narrow distributions of SWL, like that shown in the top left panel, FVR behavior decreases the response variance. Note that for a wider distribution, focal values of 0 and 10 would become prominent, and the variance would be biased upwards instead.

Each of the other panels in Figure 7 show the effect of varying the strength of FVR behavior, i.e., of the fraction of respondents who project their response onto the simplified 3-point scale of (0, 5, 10). This is accomplished by varying the parameter $\beta_N^0$ in Eq. (5), which is transformed and named the Numeracy parameter. It affects only the fraction of respondents
who are high type and is thus inversely related to the FVRI. Note that $\beta^h_{N}$, the relationship between numeracy and education, is constant for all cases shown.

For the reason just described, the top right panel shows a zero bias in the mean SWL for all values of the Numeracy parameter. By contrast, estimates of the education and income effects are strongly affected by FVR even for this case of a symmetric and centred distribution of SWL responses. The second row of Figure 7 shows biases in these coefficients resulting from a maximum likelihood ordered logit estimator. The FVR effects shift low-SWL responses upwards to “5” and high-SWL responses downwards to “5.” These contribute constructively to produce a large downward bias in the estimated effect of education on SWL.

The same is true of the bias of the income coefficient, shown in the panel on the right in the second row. The bias is also large and negative, although somewhat smaller due to the imperfect correlation ($\chi = 0.45$) between individual education, which determines numeracy in the model, and income.

Interestingly, when the FVR becomes extreme, and the majority of responses are focal values, the bias of the ordered logit coefficient for education attenuates and then even reverses sign when more than 80% of respondents use focal values. In that regime, the small numbers of “0” and “10” responses are dominant sources of the variation in the dependent variable.

The third row of Figure 7 presents results from the mixture model. The blue squares show estimates for a degenerate version of the model, in which response behaviour is restricted to high type (labeled ologit); these agree closely with the commonly-used ordered logit estimates above. As mentioned in Section 4.4, the full version of this model correctly recovers the true (synthetic) parameters for the effects of education and income on SWL. This is shown by the zero biases for the green diamonds in the third row. The model can also correctly estimate a counterfactual distribution of observations which would have obtained had FVR behavior not existed — i.e., if even low-type respondents used the full scale. This distribution of “latent” life satisfaction reports is shown, with 95% compatibility (confidence) intervals, in green in the top left panel.

Lastly, the lower left panel shows the DGP and model-estimated values of the FVRI, i.e., the number of low-type respondents among the observations. Again, the mixture model estimator correctly estimates this value.

Figure 8 has the same layout as Figure 7, but with a different value of the cutpoint offset parameter ($\alpha_H$) in the DGP. This difference has the sole effect of shifting the distribution of true SWL to the right. Now there is, as is more typical, a divergence between the mean responses of high and low types. While some low types shift their response down to “5”, a larger number are shifting up to “10”, leading to an overall upward bias of as much as 0.6 on the 11-point scale. As expected, the bias converges to zero when all respondents are high type, but grows correspondingly larger when focal value respondents (low-type) are plentiful.\footnote{Note that the distribution of high-type responses is not the same as the distribution of the true SWL in the synthetic sample, since numeracy is likely to be correlated with SWL. None of these three distributions (high-type only, low-type only, or hypothetical true SWL) is directly observable in real data, since focal value choice cannot be ruled out for those high types reporting focal values.}

The bias in the raw ordered logit coefficient for education is much larger in this case; FVR biases it downwards nearly to zero. The income coefficient is again nearly uniformly negatively biased in this scenario. The reader may recall that in this synthetic scenario, income plays the role of predictors which affect SWL but do not affect numeracy directly, even though they may be correlated with education. In real data, measured income may also be a predictive

\[20\]
Fig. 7. Estimates of synthetic data: variation of high type fraction with SWL≈5. The top left panel shows the distribution of responses in one synthetic data set split into those coming from high-numeracy respondents (blue) and those from low-numeracy respondents (red). The mixture model’s posterior estimate of the counterfactual response distribution unbiased by FVR is shown with 95% confidence intervals in green. Other panels show estimates from a range of synthetic data sets, varying only in the fraction of high-type respondents. The vertical grey bands indicate the value of N parameter used in the top left panel.
proxy for other factors causally related to numeracy and therefore FVR behavior. Therefore in real data the income bias may behave somewhat more like that of the education coefficient.

Fig. 8. Validation: variation of high type fraction with SWL≈8.5. The format is the same as Figure 7. This case differs only in having a right-shifted distribution of true (latent) life satisfaction.
Fig. 9. Estimates of synthetic data: dependence on average SWL. The format is similar to that of Figure 7 but in this case an offset to $\alpha^H$ is the synthetic data parameter varying across cases.
Figure 9 has a similar format again, but now the parameter which varies along the abscissa of the bias plots relates to the cut points $\alpha^H$. This “cutpoint offset” translates the distribution of SWL to the right, and is the same parameter that distinguishes Figure 7 from Figure 8. The following findings are confirmed or revealed. First, the bias in mean SWL can be significant even for moderate contributions of low-types. Furthermore, as a distribution of SWL shifts from low to high, this bias takes both signs and varies non-monotonically. The top right panel shows biases of $\pm 0.1$, which is nearly comparable in real data to large effects like marriage, unemployment, and doubling of income. Moreover, as the bulk of responses move from less than 5 to greater than 5 to closer to 10, the dominant effect of focal value substitutions changes sign twice, with zeros when the distribution is centred at 5 and 7.5. Real-world distributions of SWL tend to have means $\geq 3$; otherwise, another reversal, and a zero at 2.5, would also exist.

As already observed above, the bias on naive estimates of the effect of education on SWL can also be both positive or negative, depending on where the primary mass of true SWL lies. For both $2 < \text{SWL} < 5$ and $7 < \text{SWL} < 10$, low education is likely to bias reported SWL upwards, to 5 or to 10, respectively. As a result, the strongest effects on the education coefficient are negative biases, while the positive bias from $5 < \text{SWL} < 8$ dominates only for a small range among the synthetic trials shown in Figure 9.

By contrast with the education coefficient, the income coefficient is again uniformly negatively biased. The uniformity of a negative bias on the income coefficient reflects the fact that education is controlled for in the regression. Thus the residual variance in income is orthogonal to the random reassignment of responses towards focal values. This noise results in an attenuation bias for the income coefficient.

The mixture model in the third row of Figure 9 again closely reproduces the ordered logit estimates of the second row when restricted to exclude focal value response behaviour, and estimates the two SWL coefficients and the FVRI correctly when unconstrained.

5 Empirical estimates of FVR bias and FVRI

After validation of the mixture model estimator in Section 4.4, I now turn to the estimation of focal value biases in real data. The distributions of SWL for different levels of education, shown in Figure 4, indicate the strength of the focal value response problem in the Canadian population. Using the unconstrained mixture model, the role of education in supporting SWL can now be estimated, despite the strong relationship between education and the focal value bias. Table 1 shows the results of conventional, or “naive” estimates of the following simple individual-level cross-sectional OLS model for SWL:

\[ \text{SWL}_i = c + \beta^h_{\text{education}} + \beta^I_{\text{log (HH income)}} + \varepsilon_i \]

and its ordered logit counterpart. The estimated coefficient on education is distinctly negative. The ordered logit coefficient predicts that a university education reduces the odds ratio of a higher SWL by 13% as compared with someone who has less than a high school education. The OLS coefficient predicts the university graduate to report a life satisfaction $\sim 0.08$ lower than the least educated group, all else similar. This value is economically large, as can be seen by examining the coefficient ($\sim 0.4$) for log income from the same estimates.

To restate the mixture model outlined in Section 3, and again as the data generating process in Section 4.1, numeracy is modeled as a binary logit,
\[ \log \left( \frac{P_{\text{high type}}}{1 - P_{\text{high type}}} \right) = \beta_N^h + \beta_N^{\text{education}} \]  
and high type responses are modeled as an ordered logit,

\[ \log \left( \frac{P_{\text{SWL} \geq s}}{P_{\text{SWL} < s}} \right) = \beta_S^h \text{education} + \beta_S^{\text{log}(\text{income})} + \epsilon_S \]  
for each SWL response \( s \in (1, \ldots, 10) \) above 0. Low-type responses are modeled with the same function and same coefficients \( \beta_S \), but different cut points and with only responses \( s \in (5, 10) \) possible, other than 0.

When constrained to disallow focal value behavior, the mixture model’s estimate, shown in column (3) of Table 1, reproduces the ordered logit values. However, when the full model is estimated, a significantly positive value (\( \sim 0.08 \)) is found for the education coefficient on SWL. Its interpretation is that, after controlling for income, a university graduate has 2.5% higher odds ratio of reporting a higher life satisfaction than someone who has not completed high school.

The bias in a conventional estimate of the income coefficient is also large: the mixture model strongly rejects the naive estimated value of \( \sim 0.41 \), in favor of a value of \( \sim 0.54 \). This represents a 30% difference in the most important value in the field of the economics of happiness. The correction amounts to a 21% difference in the odds ratio for a given income change.

6 Applications

An enormous range of empirical papers estimating models of life satisfaction and other extended-Likert-like scales could be revisited in light of the significant biases identified above. Naturally, those focusing on effects of income, education, and age, and those which particularly address populations with low levels of numeracy, especially invite reanalysis. Here I reproduce estimates from just two papers to exemplify the important changes that may result from such analysis, and to show that the surprising but highly reproduced “paradox” of negative benefits to education may generally be resolved by the cognitive mixture model.

6.1 Clark and Oswald (1996)

The first of these papers, with over 1200 citations, is a relatively early contribution in the modern study of relative income concerns but also prominently points out the anomalously low estimated returns to wellbeing from education (Clark and Oswald, 1996). It was also recently cited as one of 11 studies in the major accumulated evidence on the life satisfaction benefits from additional education (Clark et al., 2019, see Annex 3a). In fact, the paper uses data from the BHPS prior to its inclusion of the life satisfaction question, so it uses instead responses to 7-point satisfaction with income and satisfaction with job questions.

Clark and Oswald (1996) did not examine the distributions of these subjective response variables according to their study’s four categories of formal educational attainment.\(^{14}\) However, doing so reveals a dramatic focal value response behavior which diminishes with education

\(^{14}\)The description from Clark and Oswald (1996) reads: “Table 5 contains two ordered probits, in each of which three dummies for educational attainment are included as well as a control for income. The dummies
Table 1. Estimates of mixture model on CCHS data. The first three columns show conventional, “naive” estimates of a model explaining life satisfaction with just two predictors, a measure of educational attainment on a 4-point scale, and the logarithm of household income. The first two columns are estimated using Stata, while column (3) shows estimates using the HCMC estimation of the mixture model but when the model is constrained not to allow for FVR behavior. Column (4) shows the unconstrained mixture model estimate, with significantly higher effects of education and income on life satisfaction.
Fig. 10. Distributions of satisfaction with job, overall, and with income for different categories of educational attainment.

(Figure 10). The income satisfaction distribution is wider and more central (i.e., near “4”) than that of job satisfaction, and features unmistakable evidence of all three focal values (1, 4, and 7) for groups with lower academic certifications. The same three focal values are less prominent but also evident, and with monotonically decreasing prominence, for the four groups’ responses to the job satisfaction question.

Table 2 shows estimates of several models for overall satisfaction with job, comprising conventional estimation approaches along with a cognitive mixture model in which educational attainment and education are used as proxies for numeracy and allowed to predict focal value behavior. Column (1) shows raw coefficient estimates of an ordered probit model, nearly reproducing the published values and retained sample size (4730 in all my estimates) of the main estimate in Clark and Oswald (1996, Table 5). For completeness, columns (2) and (3) show conventional OLS and ordered logit estimates of the same model. In all cases, academic attainment is strongly predictive of lower satisfaction after adjusting for log of household income. The implied effect is enormous. As compared with someone with primary education only, an advanced high school graduate (A-levels) is less satisfied with their job by as much as they would be with a 3-fold decrease in wage. As already shown in Figure 10, even the raw mean satisfaction is decreasing across the first three education groups. Clark and Oswald are for a college degree, advanced high school (A-level approximately), and intermediate high school (O-level approximately). The omitted category is for no or low qualifications. These four categories are for achieved paper certificates and not merely for years of schooling.”

15 The coefficient shown on log income (.016) strongly disagrees with the published value (.50) in Clark and Oswald (1996). Upon contacting the authors, it was determined that a typo in production of the original work resulted in a reporting of 0.50 rather than the estimated 0.05 for this coefficient (personal communication, Andrew Clark, 2021). This error has not been previously reported. Because of the error, the authors did not address the surprisingly low coefficient on log of household income. The set of regional, health, race, industry, and occupation dummies are excluded in Table 3 because the exact definitions from the 1996 work are not available.

16 The coefficients on log job hours and on A-levels education are nearly identical, implying that, having already adjusted for income, a unit log, or factor ∼2.7, increase in hours worked predicts a similar change to job satisfaction as does the educational attainment.
(1996) speculate that their findings of low satisfaction of the higher educated may be related to a recent recession that particularly hit the middle class in the UK, but also cite several earlier studies which corroborate the negative or negligible benefits from education on job satisfaction.

Equally surprising in these results is the nil effect of income on job satisfaction. The 95% confidence interval for the coefficient of log income in column (3) is $-0.10$ to $+0.13$, with the upper limit implying that a doubling of income would increase the odds of a higher satisfaction response by less than 10%.

Column (4) simply shows that the cognitive mixture model reproduces an ordered logit estimate when focal value behavior is turned off, while my key result lies in Column (5). When focal value behavior is accounted for, the income coefficient increases by a factor of 10 and the strongly negative coefficients on education flip sign. Estimates of other coefficients remain statistically unchanged. Both formal education and reported income prove significant in predicting numeracy, i.e., focal value behavior. The estimated fraction of respondents, overall, who restricted their answers to focal values is 25%. The model also estimates a significant bias to the mean reported job satisfaction, from a latent value of 5.1 which would have obtained had all respondents used the full scale, to the observed value of 5.5.

Table 3 parallels Table 2 but relates to the other column in Clark and Oswald (1996)’s Table 5 — an estimate for satisfaction with pay rather than with the job overall. In this case, increased income is a strong predictor of satisfaction even in naïve estimates. On the other hand, higher education again strongly predicts lower satisfaction, after adjusting for household income, in conventional models. This may make some sense if the primary effect of education in this context is to set expectations about pay. In any case, the significantly negative coefficients for educational attainment are fully eliminated by accounting for FVR, as shown in Column (5) of Table 3. At the same time, the predicted effect of household income on satisfaction increases significantly as compared with the naïve model. A doubling of income now predicts more than a doubling of the odds of responding with a higher level of satisfaction.

### 6.2 First Nations and Métis in Canada

Next I pick on my own prior work by re-examining a paper which reported an anomalously low benefit of income for a sample of Indigenous (First Nations and Métis) peoples in Canada (Barrington-Leigh and Sloman, 2016). In addition to estimating a negative effect of income on life satisfaction, we found an average life satisfaction among indigenous respondents that was equivalent to that of the general population, despite the stark objective challenges faced by the former groups, including disproportionate levels of discrimination and socioeconomic disadvantage with respect to the rest of the Canadian population. Barrington-Leigh and Sloman (2016) suggest as a possible interpretation that total income is not well measured by the standard income question for this group, but remain “cautious and skeptical” about the data overall.

This case study is interesting not only because of the importance of being able to use life satisfaction data across diverse cultural and economic circumstances, but also because it will demonstrate the use of the mixture model on a small sample. The data come from two Canadian surveys: the national Equality, Security and Community survey (ESC, $N = 3725$) and its follow-up small sample of on- and off-reserve First Nations and Métis peoples in the Canadian Prairies (Aboriginal ESC, $N = 446$). As can be seen in the first panel of Figure
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Significance: 0.1%† 1%⋆ 5% 10%+

Table 2. Estimates of job satisfaction in BHPS
## Table 3. Estimates of satisfaction with pay in BHPS

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Significance: 0.1%† 1%* 5% 10%+

Table 3. Estimates of satisfaction with pay in BHPS.
Fig. 11. Life satisfaction of Indigenous Canadians (left panels) and the whole population (right panels). The Aboriginal ESC is a distinct sample from the General ESC, while the panel labeled “Aboriginal GSS” is simply a subset of the full GSS sample.

11, an “enhancement” at SWL=10 in the Aboriginal ESC sample makes it the modal response value and may go some way to explaining the high mean reported SWL. Indeed, this is likely the first report of a SWL distribution with such a dominant response at its top value. On the other hand, respondents also gave plenty of 7s, 8s, and 9s, each with higher frequency than SWL=5. Below I use the cognitive mixture model to assess how much this distribution might be biased by FVR, and whether the anomalous estimates in Barrington-Leigh and Sloman (2016) can be reversed.

The first column of Table 4 shows a conventional ordered logit estimate of 10-point life satisfaction of the Indigenous sample. The education variable is measured on ten steps ranging from no primary school to a PhD or professional degree. Once again, and despite a sample size of only 446, a significantly negative coefficient on education shows that, after adjusting for income, those with higher education report lower life satisfaction. In addition, the coefficient on log household income is estimated to be most likely negative, with a 95% confidence interval between $-0.39$ and $+0.08$.

The second column reports the estimate of a cognitive mixture model. Education strongly predicts numeracy, i.e., use of the full response scale. Most importantly, both anomalies in the earlier analysis are again resolved when FVR is taken into account: the negative education coefficient is replaced by one with a confidence interval between $-0.06$ to $+0.18$ and the negative income coefficient becomes significantly positive.

In order to address the surprisingly high average life satisfaction reported by Indigenous respondents, pooled estimates of the Canada-wide respondents and the First Nations/Métis sample are shown in Columns (3) and (4) of Table 4. Adjusting for income and education, the Indigenous respondents report 0.30 higher life satisfaction than non-Indigenous. When FVR is accounted for (Column 3), this situation is reversed, with Indigenous respondents reporting −0.31 lower life satisfaction than others with similar income and education. In this model,
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<td>log likelihood</td>
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<td>−817</td>
<td>−7409</td>
<td>−7320</td>
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</table>

Table 4. Estimates of life satisfaction of Indigenous Canadians

education, income, and on-reserve status are all allowed to predict FVR behavior. For the pooled sample, the mixture model estimates a 30% higher income coefficient and essentially flips the sign on the significantly negative education coefficient of the naive model.

7 Discussion and conclusion

Clark and Oswald (1996) write “Counter to what neoclassical economic theory might lead one to expect, highly educated people appear less content. The effect is monotonic and well-defined.” This contradiction with neoclassical economic theory has generally held up to subsequent analysis over two decades but is resolved with a model which takes into account an obvious empirical feature of the response function.

Income effects have been a focus in the study of well-being in economics since the field’s inception, and an enormous literature exists around the magnitude of the income coefficient (e.g., Easterlin, 1974; Deaton, 2008; Clark, Frijters, and Shields, 2008; Dolan, Peasgood, and White, 2008; Easterlin, 1995, 2013; ?; Kapteyn, van Praag, and van Herwaarden, 1978; Luttmer, 2005; Senik, 2005; Van Praag and Kapteyn, 1973). In this context, the above findings from a proper treatment of focal response bias are quite stark. Almost every study of SWL includes an estimate of the income effect, and typically other influences on life satisfaction are quantified in terms of their income “compensating differential,” i.e., the ratio between a coefficient of interest and the income coefficient. Thus, the large corrections estimated here for
the income coefficient not only indicate that material supports are quantitatively more effective for raising human well-being than the literature has shown so far, but also have implications for the estimated marginal benefits of all the other, especially social, dimensions of life; namely, they are slightly less strongly hard-coded into our nature than the overwhelmingly large values that are a stylized fact in this field.

It should be noted that in relatively high-SWL developed countries where the mass of respondents’ “true” life satisfaction lies between 7 and 8, the opposing biases to mean SWL associated with focal values of 5 and 10 might largely cancel out. By contrast, the impacts of these two focal values can work constructively when it comes to biasing the coefficient of education on SWL.

More generally, effects will differ across countries according to average SWL levels and according to the income and education distribution. In addition, the approaches described here should be more sensitively able to discern any international differences in the tendency to use focal values, so that differences in education systems or more cultural drivers of focal value can be incorporated into international comparisons of SWL. The new methodology of flexibly modeling each possible SWL response so as to allow for non-ordinal relationships between them, carried out here using multinomial logit, is a good starting point for better evaluating such response biases. For instance, previous investigations into the possibility of cultural differences in the use of the SWL scale have not taken this approach in order to look for cultural norms. Despite the general evidence of good comparability of SWL patterns across cultures, it may still be possible to identify response biases towards central values or away from “extreme” values. That work represents a separate investigation.

An analysis of panel data will also be important, and individual fixed effects are not captured in the analysis conducted here. Preliminary analysis of panel data with a 5-point scale for SWL, treating values 1, 3, and 5 as focal values, shows that the probability of SWL changing from the middle value is decreasing in education. Traditional 1st-differences approaches for panel fixed effects are invalid because, for instance, the dependence of the 3 → 4 transition is not the mirror of the 4 → 3 transition.

With respect to the question of survey and questionnaire design, attention to variation in cognitive capacity of respondents speaks to the value of letting respondents choose and even identify their own resolution. Open-ended graphical scales may be one means to accomplish this, but further research into ways to elicit a statement of precision from respondents would be valuable. The potential for creativity and innovation is high, given the increasing availability of technology during an interview.

Depending on one’s perspective, the present findings may be taken as a warning of how difficult it would be to realize the most ambitious implementations of SWL as a guide to policy (Barrington-Leigh and Escande, 2018; Barrington-Leigh, 2016; Frijters et al., 2020; Frijters and Krekel, 2021; Happiness Research Institute, 2020; Barrington-Leigh, 2021) or, conversely, as another example of the general robustness of SWL inference to potential flaws inherent in its cognitive complexity, and possibly even a defense of the magnitudes of estimated effects that have become so reproducible in study after study. I take away both of these messages.

\[17\] In South Korea, with a high and increasing SWL, almost no one responds with the top response option.
References


A Estimation

The cognitive model Eq. (3) can be estimated using the observed responses $S_i$ and characteristics $x_i$ of individuals $i$. The estimation objective is to find the marginal effects associated with $\beta_S$, and to compare these with those derived from the canonical regressions which are naive to the preferential use of focal values. With or without the constraints on $\{\alpha^L\}$, the unknown parameters can be estimated by the maximum likelihood method, i.e. by maximising

$$L(\beta_N, \beta_S, \alpha^H, \alpha^L | S, z, x) = \sum_i \ln \Pr(S_i | x_i)$$

(11)

where the first sum is over individuals $i$ with observed response $S_i$ and characteristics $x_i$, and where $1(S_i = s) \equiv 1$ when $S_i = s$ and 0 otherwise.

Nothing guarantees convexity of the objective function, so a “hopping” algorithm, for instance as implemented in Python’s SciPy suite, can be used to search for a global maximum. Bootstrapping of the data is then used both to assess confidence due to sampling and to ensure that the hopping algorithm is robustly attaining a global optimum. Alternatively, a Markov chain Monte Carlo procedure using a Bayesian estimation framework may provide better efficiency and coverage of the sample space. This latter approach was used for all mixture model estimates reported in this paper. Section A.1 provides derivations of gradients and Hessians for computation of the log likelihood. Section A.2 outlines the priors used for Bayesian estimates.

A.1 Computation

For computational efficiency, it is useful to compute the gradient and Hessian of the objective function. I use the following shorthand, below:

$$P_s \equiv \Pr(s | x) = P_HP_S^H + [1 - P_H] P_S^L$$

$$P_S^H \equiv \Pr(s | x, \text{high}) = \Phi_S^H - \Phi_{S-1}^H$$

$$P_S^L \equiv \Pr(s | x, \text{low}) = \Phi_S^L - \Phi_{S-1}^L$$

$$\Phi(\cdot) = \Phi_N(\cdot) = \Phi_S(\cdot)$$

$$\Phi_S^H \equiv \Phi(\alpha^H_S - x'\beta_S)$$

$$\Phi_S^L \equiv \Phi(\alpha^L_S - x'\beta_S)$$

$$P_H = \Phi_N \equiv \Phi(z'\beta_N) = 1 - \Phi(-z'\beta_N)$$

$$\mathbb{1}_{a,b} \equiv \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

and $D(\cdot), \partial, \nabla$ denote total derivative, partial derivative, and gradient operators.

The log likelihood is thus written

$$L = \sum_i \ln P_s$$
where $s$ refers to $S_i$, the response of individual $i$.

Also, note that for the logistic CDF, $\Phi(\xi) = \frac{1}{1 + e^{-\xi}}$, we have:

$$D(\log \Phi) = 1 - \Phi$$

and

$$D(\Phi) = \Phi [1 - \Phi]$$

and

$$D^2(\Phi) = \Phi - 3\Phi^2 + 2\Phi^3$$

Starting with the top-level notation, we have the gradient with respect to parameter $k$

$$\partial_k \mathcal{L} = \sum_i \frac{1}{P_s} \partial_k P_s$$

and thus the Hessian matrix

$$\partial_j \partial_k \mathcal{L} = \sum_i \left[ \frac{-1}{P_s^2} \partial_j P_s \partial_k P_s + \frac{1}{P_s} \partial_j \partial_k P_s \right]$$

### A.1.1 Gradient

All derivatives below refer to a single respondent. The subscript $s$ refers to the particular value reported by respondent $i$. The gradient $\partial P_s$ can be expressed in general as

$$\partial P_s = \begin{bmatrix} P_s^H - P_s^L \end{bmatrix} \partial P_H + P_H \partial P_s^H + \begin{bmatrix} 1 - P_H \end{bmatrix} \partial P_s^L$$

Considering the four groups of parameters in the parameter vector $v = \begin{bmatrix} \beta_N' & \beta_S' & \alpha_H' & \alpha_L' \end{bmatrix}$, we can express the components of the gradient (Eq. (12)) separately as follows, based on the limited parameter dependencies $P_H = P_H(\beta_N)$; $P_s^H = P_s^H(\beta_S, \alpha_H)$; and $P_s^L = P_s^L(\beta_S, \alpha_L)$:

$$\nabla_{\beta_N} P_s = \begin{bmatrix} P_s^H - P_s^L \end{bmatrix} \nabla_{\beta_N} P_H$$

$$= \begin{bmatrix} P_s^H - P_s^L \end{bmatrix} \Phi_N [1 - \Phi_N] \mathbf{z}$$

$$\nabla_{\beta_S} P_s = P_H \nabla_{\beta_S} P_s^H + \begin{bmatrix} 1 - P_H \end{bmatrix} \nabla_{\beta_S} P_s^L$$

$$= \begin{bmatrix} \Phi_S \end{bmatrix} \nabla_{\beta_S} P_s^H$$

$$\nabla_{\alpha_H} P_s = P_H \nabla_{\alpha_H} P_s^H$$

$$\nabla_{\alpha_L} P_s = \begin{bmatrix} 1 - P_H \end{bmatrix} \nabla_{\alpha_L} P_s^L$$

In turn:

$$\nabla_{\beta_S} P_s^H = - \Phi_S^H [1 - \Phi_S^H] \mathbf{x}$$

$$+ \Phi_S^H [1 - \Phi_S^H] \mathbf{x}$$

$$= \mathbf{x} \left[ -D(\Phi_S^H) + D(\Phi_S^H) \right]$$
and similarly for $H$ replaced by $L$. Elements of $\nabla_{\alpha_{H}} P_{s}^{H}$ are as follows\textsuperscript{18}:

$$\nabla_{\alpha_{H}} P_{s}^{H} = + \Phi_{S}^{H} [1 - \Phi_{S}^{H}] \mathbb{I}_{s}^{H} - \Phi_{S-1}^{H} [1 - \Phi_{S-1}^{H}] \mathbb{I}_{s-1}^{H} = + D (\Phi_{S}^{H}) \mathbb{I}_{s}^{H} - D (\Phi_{S-1}^{H}) \mathbb{I}_{s-1}^{H}$$

and similarly for $H$ replaced by $L$.

### A.1.2 Hessian

Using again a functional dependence on the vector $v = \left[ \beta'_{N} \beta'_{S} \alpha'_{H} \alpha'_{L} \right]$, the non-zero terms of the Hessian are as follows:

$$\begin{pmatrix}
\frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \beta_{k}} \mathcal{L} & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \beta_{k}} \mathcal{L} & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \alpha_{k}} P_{s}^{H} & \frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \alpha_{k}} P_{s}^{L} \\
0 & 0 & \frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \alpha_{k}} P_{s}^{L} & 0
\end{pmatrix}$$

Expanding the terms of the Hessian:

$$\frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \beta_{k}} P_{s} = \frac{\partial}{\partial \beta_{j}} \frac{\partial}{\partial \beta_{k}} \left( [P_{s}^{H} - P_{s}^{L}] D (\Phi_{N}) z_{k} \right) = [P_{s}^{H} - P_{s}^{L}] D^{2} (\Phi_{N}) z_{j} z_{k}$$

\textsuperscript{18}If we define an $N \times (S - 1)$ sparse matrix, $\mathbb{1}_{S}$, with 1s in columns corresponding to each observation’s $y$-value, and another, $\mathbb{1}_{S-1}$ with 1s in columns corresponding to one less than each observation’s $y$-value, we can write the above in terms of a $1 \times N$ row vector $\Phi_{S}^{H}$ of values for each respondent.

$$D_{\alpha_{H}} P_{s}^{H} = \Phi_{S}^{H} \circ \left[ 1 - \Phi_{S}^{H} \right] \circ \mathbb{1}_{S}$$

This way, all the components of $\nabla P_{S}$ can be put into a matrix with rows corresponding to observations, which is convenient for computation.
\[
\frac{\partial}{\partial \alpha_j^H} \frac{\partial}{\partial \beta_k^N} P_s = \frac{\partial}{\partial \alpha_j^H} \left( [P_s^H - P_s^L] D (\Phi_N) z_k \right)
\]
\[
= D (\Phi_N) z_k \frac{\partial}{\partial \alpha_j^H} (P_s^H)
\]
\[
= D (\Phi_N) z_k \frac{\partial}{\partial \alpha_j^H} \left( H (\alpha_s) 1_s^H - D (\Phi_{S-1}^H 1_s^H) \right) D (\Phi_N) z_k
\]
(20)

and

\[
\frac{\partial}{\partial \alpha_j^L} \frac{\partial}{\partial \beta_k^N} P_s = \frac{\partial}{\partial \alpha_j^L} \left( [P_s^H - P_s^L] D (\Phi_N) z_k \right)
\]
\[
= D (\Phi_N) z_k \frac{\partial}{\partial \alpha_j^L} (-P_s^L)
\]
\[
= D (\Phi_N) z_k \frac{\partial}{\partial \alpha_j^L} \left( H (\alpha_s) 1_s^H - D (\Phi_{S-1}^H 1_s^H) \right) D (\Phi_N) z_k
\]
(21)

and same for L. Next, the cross terms between elements of $\beta_S$

\[
\frac{\partial}{\partial \beta_j^H} \frac{\partial}{\partial \beta_j^N} P_s = \frac{\partial}{\partial \beta_j^H} \left( P_H \frac{\partial}{\partial \beta_k^N} P_s^H + [1 - P_H] \frac{\partial}{\partial \beta_k^N} P_s^L \right)
\]
\[
= x_k P_H \frac{\partial}{\partial \beta_j^H} \left( -D (\Phi_S^H) + D (\Phi_{S-1}^H) \right)
\]
\[
= x_k P_H \left( -D^2 (\Phi_S^H) 1_s^H + D^2 (\Phi_{S-1}^H) \right)
\]
and lastly, cross-terms between $\beta_S$ and $\alpha_S$:

\[
\frac{\partial}{\partial \alpha_j^L} \frac{\partial}{\partial \beta_k^N} P_s = \frac{\partial}{\partial \alpha_j^L} \left( P_H \frac{\partial}{\partial \beta_k^N} P_s^H + [1 - P_H] \frac{\partial}{\partial \beta_k^N} P_s^L \right)
\]
\[
= \frac{\partial}{\partial \alpha_j^H} \left( P_H \frac{\partial}{\partial \beta_k^N} P_s^H \right)
\]
\[
= x_k P_H \frac{\partial}{\partial \alpha_j^H} \left( -D (\Phi_S^H) + D (\Phi_{S-1}^H) \right)
\]
\[
= x_k P_H \left( -D^2 (\Phi_S^H) 1_s^H + D^2 (\Phi_{S-1}^H) \right)
\]
\[ \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \alpha_k} P_S = P_H \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \alpha_k} P_H^S \]

\[ = P_H \left[ \mathbb{1}_{j,k,s}D^2(\Phi^H_S) - \mathbb{1}_{j,k,s-1}D^2(\Phi^H_{S-1}) \right] \]

and similarly for low types but with \( 1 - P_H \) replacing \( P_H \), where the \( \mathbb{1} \) is for the appropriate type.

### A.1.3 Constraints

For Lagrangian-based constrained optimization, the constraints on \( \alpha_H \) and \( \alpha_L \)

\[ 0 < \alpha_H^{i+1} - \alpha_H^i < \infty \]

can be expressed in a matrix

\[ \begin{pmatrix} 
\ddots & 0 & 0 & 0 \\
0 & -1 & 1 & \\
0 & -1 & 1 \\
0 &
\end{pmatrix} \]

### A.2 Bayesian priors

For the Bayesian estimations carried out in this work, the following priors were used. All coefficients \( \beta_N \) and \( \beta_S \) are given normal priors with mean zero and standard deviation 3. Given that typical estimates have \( |\beta| < 1 \), these are considered to be weak priors and to accommodate estimates of either sign. Cut points are initially assumed to be distributed with induced Dirichlet priors (Sethuraman, 1994).